same parts BAX, HXA, less than two right angles, must at length meet toward those parts in one and the same point H.

I premise fourthly : likewise will be no doubt over the truth of the preceding hypothetical assumption, if those later external angles YHD, YDP and so any other succeeding ones, either always are equal to the preceding external angle YBD, or at least always will be not so much less but that any one of them always will be greater than any little designated acute angle K. For, this holding, it is manifest that this XY in that however great motion of its toward the parts of the point Z, never will cease to cut the aforesaid AY; which assuredly (from the preceding note) is sufficient for establishing the controverted postulate.

Solely therefore remains, that a certain adversary may say those external angles at greater and greater distance from that point A may become always less without any determinate limit.

But thence would follow, that that XY in that motion of its above the straight AZ would at length meet AY in a certain point P without any angle with the segment PY, so that indeed a segment of the two straights APY, and XPY would be in this way common.

But this is evidently repugnant to the nature of the straight line. [The possibility that P may be a point at infinity is here overlooked.]

But if indeed to anyone may seem less opportune the obtuse angle at that point X toward the parts of the point A, it may easily be supposed right ; so that indeed (in the motion of the aforesaid XY at angles always right above the straight AZ) more manifestly may appear that the single points of that XY are always moved equably relatively to the basal AZ; and therefore the aforesaid XY cannot go over from a secant into a non-secant of the other indefinite AY, unless either once in some point it precisely touches it, or meets it in some point P, where it has with this AY a common segment PY; each of which I will show contrary to the nature of the straight line in P. XXXIII.

Therefore in accordance with the true idea of the straight line, must that XY, in however great distance of the point X from the point A, always meet in some point this AY. And that this indeed (however small is supposed the acute angle at the point A) is sufficient for demonstrating. against the hypothesis of acute angle, the Euclidean Postulate, will follow from P. XXVII.

[To be Continued.]

THE BOND PROBLEM.

By J. K. ELLWOOD, A. M., Colfax School, Pittsburg, Pennsylvania.

What should an investor pay for one 7 per cent. $100. bond to run 20 years, interest payable semi-annually, in order to realize 8 per cent. per annum, payable semi-annually ?

Let X=the price paid; R=4%, the semi-annual rate the investor realizes; t=the whole number of interest payments; r=3½%, the rate the bond draws semi-annually; v=\$100.

Besides the interest, the investor gains $v-x$, which will be due in ½t years. To liquidate both of these by equal payments requires each semi-annual payment to include the interest (rv) and such portion of the discount ($v-x$) as would, compounded semi-annually at R%, amount to $v-X$ in ½t years.

Let y be such a sum; then

$y(1+R)^{t-1}$=amount of 1st installment at end of ½t years.
$y(1+R)^{t-2}=$ " " 2nd " " " " "
$y(1+R)=$ " " $(t-1)^{th}$ " " " " "
$y(1+R)^0 =$ " " t^{th} " " " " "

Hence, $y[(1+R)^{t-1}+(1+R)^{t-2}+ \ldots \ldots (1+R)+1]=v-X$.

Summing the geometrical progression within the brackets, we have

$$y\left[\frac{(1+R)^t -1}{R}\right]=v--X,$$

$$\text{whence } y=\frac{R(v-X)}{(1+R)^t -1}.$$

Therefore each of the t equal payments is

$$vr+\frac{R(v-X)}{(1+R)^t -1},$$

which divided by X gives R.

$$\text{Hence, } vr+\frac{R(v-X)}{(1+R)^t -1}=RX.$$

Solve this equation for X and we have:

$$X=\frac{v(R-r)+vr(1+R)^t}{R(1+R)^t}\ldots\ldots(A).$$

In the above general equation substitute values from the problem and we have:

$$X=\frac{100(.04-.03\frac{1}{2})+3\frac{1}{2}(1\ 04)^{40}}{.04(1.04)^{40}}=\frac{\frac{1}{2}+3\frac{1}{2}\times 1.04^{40}}{.04\times 1.04^{40}}$$

The easy numerical computations are as follows:

40log 1.04=0.0170333×40=0.681332, which corresponds to 4.801.

$$\frac{\frac{1}{2}+3\frac{1}{2}\times 4.801}{.04\times 4.801}=\frac{17.3035}{.19204}=90.1036.$$

∴. $90.1036 is the price to be paid for a 7% $100 bond, interest payable semi-annually for 20 years, in order to realize 8% per annum, payable semi-annually.

The general equation (A) can be applied to the solution of the quarterly bond. In so applying it "we solve the government problem which confronted the Secretary of the Treasury when he placed the late $50,000,000 loan on the market." This problem has been admirably solved by Theodore L. DeLand, the distinguished Examiner of the U. S. Civil Service Commission, first by algebraic analysis in THE AMERICAN MATHEMATICAL MONTHLY, and later by using the Calculus of Finite Differences. The latter solution was issued under cover of the *Mathematical Magazine*, January, 1895.

Secretary Carlisle desired to sell 10-year 5% $100 bonds, interest payable quarterly, at a price that would enable the purchaser to realize 3%, interest payable quarterly.

Using these data, we have $R=\frac{5}{4}\%$, $r=1\frac{1}{4}\%$, $t=40$. Substituting values, equation (A) becomes:

$$x=\frac{100(.00\frac{5}{4}--.01\frac{1}{4})+1\frac{1}{4}(1.0075)^{40}}{.0075\times 1.0075^{40}}=\frac{1\frac{1}{4}\times 1.0075^{40}-\frac{1}{4}}{.0075\times 1.0075^{40}}.$$

40log $1.0075=0.0032451\times 40=0.129804$, which corresponds to 1.34835.

$$\frac{1\frac{1}{4}\times 1.34835-\frac{1}{4}}{.0075\times 1.34835}=117.223.$$

∴. $117.22\frac{3}{10}$ is a just price for the bonds mentioned.

Problems of this kind may be solved very readily by arithmetic, as follows:

Take the first problem above. The bond yields $7. per annum, which is 8% of $87.50. This would be the price if only $87.50 were to be paid the investor at maturity. But he will receive $12.50 more, hence he must now give, in addition to the $87.50, a sum that will in 20 years at 8% compound semi-annual interest amount to $12.50.

$$\$12.50\div\$4.80102=2.6036.$$

∴. $87.50+$2.6036=$90.1036, the price.

When bonds are bought at a premium, the present value must be *deducted* from the sum that would be the price to be paid provided that sum were to be paid the investor at maturity.

Such problems are readily solved, but the arithmetician requires a very complete compound interest table to cover all cases.

The tables used by brokers give the same prices as those obtained by the methods herein set forth; but they extend only to 6% bonds to run 60 years.